Exercise 2

Find the general solution for each of the following first order ODEs:

$$xu' - 4u = x^5 e^x, \ x > 0$$

Solution

First rewrite the differential equation so that the coefficient of u' is 1.

$$u' - \frac{4}{x}u = x^4 e^x$$

This is an inhomogeneous first order linear ODE, so we can multiply both sides by the integrating factor,

$$I(x) = e^{\int -\frac{4}{x}\,dx} = e^{-4\ln x} = x^{-4},$$

to solve it. The equation becomes

$$x^{-4}u' - 4x^{-5}u = e^x.$$

Observe that the left side can be written as $(x^{-4}u)'$ by the product rule.

$$\frac{d}{dx}(x^{-4}u) = e^x$$

Now integrate both sides with respect to x.

$$x^{-4}u = e^x + C$$

Therefore,

$$u(x) = x^4(e^x + C), \ x > 0.$$